Linear Functional Analysis

Lecture 1: Introduction

Rynne and Youngson §1.1, §1.2

History

- Functional analysis is the child of the 20th century (Stefan Banach, Hilbert, Lebesgue)
- Fourier, Riemann, Lebesgue

$$x = \sum_{n=1}^{\infty} (x, e_n) e_n$$

Ingredients

- Linear algebra (vector spaces)
- Analysis (calculus)
- Measure and integration

Why?

- We want problems to have solutions
- We want the solutions to be unique
- We want to be able to calculate (approximate) the solution

$$x = \sum_{n=1}^{\infty} (x, e_n) e_n$$

Examples

- We want problems to have solutions
- We want the solutions to be unique
- We want to be able to calculate (approximate) the solution

- Solving equations:
  $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$ ...

- Existence, uniqueness, computation of Fourier series ...
  connections to solutions of differential equations ...
  eigen-values and eigen-*functions* ...
  waves sound light heat quantum physics ...

Huishoudelijke

- Blackboard
- Homework
- Rynne and Youngson
- Schedule (Gill , de Jeu)

Bryan P. Rynne, Martin A. Youngson

Linear Functional Analysis

Springer-Verlag

Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

We now briefly outline the contents of the book. In Chapter 1 we present (for reference and to establish our notation) various basic ideas that will be required throughout the book. Specifically, we discuss the results from elementary linear algebra and the basic theory of metric spaces which will be required in later chapters. We also give a brief summary of the elements of the theory of Lebesgue measure and integration. Of the three topics discussed in this introductory chapter, Lebesgue integration is undoubtedly the most technically difficult and the one which the prospective reader is least likely to have encountered before. Unfortunately, many of the most important spaces which arise in functional analysis are spaces of integrable functions, and it is necessary to use the Lebesgue integral to overcome various drawbacks of the elementary Riemann integral, commonly taught in real analysis courses. The reader who has not met Lebesgue integration before can still read this book by accepting that an integration process exists which coincides with the Riemann integral when this is defined, but extends to a larger class of functions, and which has the properties described in Section 1.3.

Example 3.39

The set of functions $\{e_n\}$, where $e_n(x) = (2\pi)^{-1/2}e^{inx}$ for $n \in \mathbb{Z}$, is an orthonormal sequence in the space $L^2_{\mathbb{C}}[-\pi, \pi]$.

$$x = \sum_{n=1}^{\infty} (x, e_n) e_n \qquad (3.5)$$

$$(f, g) = \int_X f\overline{g}\, d\mu$$

Ch. 1 Section 2: metric spaces

- M – compact, metric space
- $\mathbb{F}$ – field $\mathbb{R}$ or $\mathbb{C}$
- $C_{\mathbb{F}}(M)$ [*aka* $C(M)$] – space of continuous functions from compact metric space M to field F endowed with the uniform metric

- $C(M)$ is complete
- Stone-Weierstrass: for M subset $\mathbb{R}$, $\mathscr{P}_{\mathbb{R}}$ is dense in $C_{\mathbb{R}}(M)$